Kuwait University | Math 101 | Date: November 17, 2008
Dept. of Math. & Comp. Sci. | First Exam | Duration: 90 minutes

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions:

1. Evaluate the following limits, if they exist

   (a) $\lim_{x\to 0} \dfrac{x}{1-\sqrt{1+3x}}$ (2 pts.)

   (b) $\lim_{x\to 0} \dfrac{x+\sin x}{3x-\tan(2x)}$ (2 pts.)

2. Find the x-coordinates of the points at which the function f is discontinuous, where

$$f(x) = \begin{cases} 2x+1 & \text{, if } x < 0, \\ \dfrac{x^2-x-2}{x^2-3x+2} & \text{, if } x \geq 0. \end{cases}$$

   Classify the types of discontinuity of f as removable, jump, or infinite.

   (4 pts.)

3. Find the vertical and horizontal asymptotes, if any, for the graph of

$$f(x) = \frac{|x|(x-1)}{x^2+x}.$$ (4 pts.)

4. (a) State the Intermediate Value Theorem.

   (b) Let f be continuous on $[0,1]$ such that $f(0) = a$ and $f(1) = b$, where $a > 0$ and $b < 1$. Show that the equation $f(x) = x$ has at least one root in $(0,1)$.

   (4 pts.)

5. Let f be a function such that $f'(1) = 3$. Find

$$\lim_{x\to 1} \frac{f(x)-f(1)}{x^2-1}.$$ (3 pts.)

6. Let $f(x) = x^{5/3} - 5x^{2/3} + 1$. Find the x-coordinates of the points at which the graph of f has:

   (a) a horizontal tangent line,

   (b) a vertical tangent line (3 pts.)

7. Find $\dfrac{dy}{dx}$, where $y = \sec^2(3x) + \dfrac{x}{x^2-1}$ (3 pts.)

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1. (a) $\lim_{x\to 0}\frac{x}{1-\sqrt{1+3x}}\times\frac{1+\sqrt{1+3x}}{1+\sqrt{1+3x}}=\lim_{x\to 0}\frac{x\left[1+\sqrt{1+3x}\right]}{-3x}=\boxed{-\frac{2}{3}}$

   (b) $\lim_{x\to 0}\frac{x+\sin x}{3x-\tan(2x)}=\lim_{x\to 0}\frac{x(1+\frac{\sin x}{x})}{x[3-\frac{\tan(2x)}{x}]}=\frac{\lim_{x\to 0}(1)+\lim_{x\to 0}\frac{\sin x}{x}}{\lim_{x\to 0}(3)-2\lim_{2x\to 0}\frac{\tan 2x}{2x}}=\frac{1+1}{3-2(1)}=\boxed{2}$

2. $\lim_{x\to 0^-}f(x)=1,\ f(0)=-1,\ \lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}\frac{(x+1)(x-2)}{(x-1)(x-2)}=-1\implies f$ has a *jump discontinuity* at $x=0$.

   $\lim_{x\to 1^\pm}f(x)=\lim_{x\to 1^\pm}\frac{(x+1)(x-2)}{(x-1)(x-2)}=\pm\infty\implies f$ has an *infinite discontinuity* at $x=1$.

   $\lim_{x\to 2}f(x)=\lim_{x\to 2}\frac{(x+1)(x-2)}{(x-1)(x-2)}=3\implies f$ has a *removable discontinuity* at $x=2$.

3. $f(x)=\frac{|x|(x-1)}{x^2+x}=\frac{|x|(x-1)}{x(x+1)}$. $\boxed{\lim_{x\to 0^\pm}f(x)=\mp 1}$ $\lim_{x\to -1^\pm}f(x)=\boxed{\pm\infty}\Rightarrow\boxed{x=-1}$ is $V.A$ for the graph of f.

   $\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{x^2\left(1-\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x}\right)}=1$ & $\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-x^2\left(1-\frac{1}{x}\right)}{x^2\left(1+\frac{1}{x}\right)}=-1\Rightarrow\boxed{y=1\text{ and }y=-1}$ are $H.A$ for the graph of f.

4. (b) Let $F(x)=f(x)-x$. F is continuous on $[0,1]$, $F(0)=f(0)+0=a>0$ and $F(1)=f(1)-1=b-1<0$. From the Intermediate Value Theorem $\exists\ c\in(0,1)$ such that $F(c)=0$, *i.e.*, $f(c)=c$.

5. $\lim_{x\to 1}\frac{f(x)-f(1)}{x^2-1}=\lim_{x\to 1}\frac{f(x)-f(1)}{(x-1)(x+1)}=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}\times\lim_{x\to 1}\frac{1}{x+1}=f'(1)\times\frac{1}{2}=\boxed{\frac{3}{2}}$

6. $f'(x)=\frac{5}{3}x^{\frac{2}{3}}-\frac{10}{3\sqrt[3]{x}}=\frac{5(x-2)}{3\sqrt[3]{x}}$. f has a horizontal tangent line at $x=2$ $(f'(2)=0)$.

   f has a vertical tangent line at $x=0$, since f is continuous at $x=0$, $\lim_{x\to 0^\pm}f'(x)=\mp\infty$.

7. $y=(\sec 3x)^2+\frac{x}{x^2-1}$.

   $y'=2\sec 3x\left[3\sec 3x\tan 3x\right]+\frac{-1-x^2}{(x^2-1)^2}=6(\sec 3x)^2\tan 3x-\frac{x^2+1}{(x^2-1)^2}$.